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THEORETICAL AND PRACTICAL ASPECTS OF SINGULARITY AND EIGENMODE EXPANSION METHODS

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ABSTRACT

Foundations of singularity and eigenmode expansion methods and some practical questions related to the theory are discussed.

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SIGNIFICANCE AND EXPLANATION

In recent years a vast amount of literature has been written on singularity and eigenmode expansion methods (SEM and EEM). The main practical purpose is to provide a theory which allows us to identify a flying (or stationary target) from the observed transient field scattered by the target. This problem is discussed in the present report. We describe 1) the basic starting points of the engineering approach to the problem, and the extent to which they are consistent from the mathematical point of view; 2) the material which has been rigorously established in the field by the writer and other authors; 3) the important points in practice, and 4) the unsolved mathematical problems in the field. The results presented may probably save efforts by other scientists and engineers since the results demonstrate the kind of research that can be used in practice and the kind which is of mostly theoretical interest.

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THEORETICAL AND PRACTICAL ASPECTS OF SINGULARITY AND EIGENMODE EXPANSION METHODS

A. G. Ramm*

§1. Introduction.

Vast literature has been written on singularity and eigenmode expansion methods during the last decade. Engineers and physicists stimulated interest in the subject (see [1], [2], [3] and references given in this review). Mathematical analysis of the problems was initiated in [4], [5] and was pushed considerably further by M. Agranovich (see [1]). Nevertheless many questions in the theory are open and of considerable interest to engineers and mathematicians. The purpose of this paper can be summarized as follows: We are going to explain in a simple way the principal features of the singularity and eigenmode expansion methods and to formulate explicitly 1) what has been used by engineers with proof, 2) what is important for practice, 3) what has been rigorously established and 4) what are the unsolved mathematical problems in this field.

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The main results obtained in scalar wave scattering theory were generalized to electromagnetic wave scattering without much difficulty. That is why and also for simplicity we restrict ourselves by the presentation of the theory for scalar wave scattering.

§2. What are singularity and eigenmode expansion methods (SEM and EEM)?

1. What is EEM?

Consider the problem

$$(\nabla^2 + k^2)u = 0 \quad \text{in} \quad \Omega = \mathbb{R}^3 \ D, \quad k^2 > 0 , \tag{1}$$

$$u|_{\Gamma} = 0 \quad , \tag{2}$$

$$u = u_0 + v = \exp\{ik(\nu , x)\} + v \tag{3}$$

and v satisfies the radiation condition

$$\frac{\partial v}{\partial |x|} - i k v = o(|x|^{-1}) \quad \text{as} \quad |x| \to \infty , \tag{4}$$

Here Γ is the smooth surface of a finite obstacle D.

If we look for a solution of (1) - (4) in the form

$$u = u_0 + \int_{\Gamma} \frac{\exp(ikr_{xt})}{4\pi r_{xt}} f(t)dt, \quad r_{xt} = |x - t| , \tag{5}$$

then

$$A(k)f \equiv \int_{\Gamma} \frac{\exp(ikr_{st})}{4\pi r_{st}} f(t)\, dt = -u_0(s), \quad s \in \Gamma. \tag{6}$$

The EEM method can now be explained as follows. Suppose that the set of eigenvectors of operator $A(k)$

$$A(k)\phi_j = \lambda_j(k)\phi_j, \quad j = 1,2,\ldots.. \tag{7}$$

forms a Riesz basis of $L^2(\Gamma) \equiv H$. This means that any $g \in H$ can be expanded in the series

$$g = \sum_{j=1}^{\infty} g_j \phi_j, \tag{8}$$

and

$$c_1 \|g\|^2 \le \sum_{j=1}^{\infty} |g_j|^2 \le c_2 \|g\|^2, \quad c_1 > 0, \tag{9}$$

where $\|g\|$ is the norm in $L^2(\Gamma)$. The inequality (9) substitutes the Parseval equality for orthonormal bases. A complete system in H does not necessarily form a basis of H (example: $H = L^2[0,1]$, the system $\phi_j(x) = x^j$, $j = 0,1,2,\ldots$: Not every $g \in L^2[0,1]$ can be expanded in the series $g(x) = \sum_{j=0}^{\infty} g_j x^j$.

If the assumption made is true, then equation (6) can be solved by the Picard formula

$$f(s) = -\sum_{j=1}^{\infty} \frac{u_{0j}}{\lambda_j(k)} \phi_j(s) \tag{10}$$

where the coefficients are uniquely defined by the equality

$$u_0 = \sum_{j=1}^{\infty} u_{oj}\phi_j \qquad (1)$$

This method of solution of the scattering problem (1)-(4) is called EEM. It was used without mathematical analysis by engineers [1], [3]. The questions, which immediately arise, can be formulated as follows: 1) Is it true that the nonselfadjoint operator $A(k)$ has eigenvectors (e.g. Volterra operator has no eigenvectors); 2) Is it true that the set of eigenvectors of $A(k)$ forms a Riesz basis of H; 3) suppose that the set of eigenvectors ($\equiv$ eigensystem) of $A(k)$ does not form a Riesz basis of H. Is it true that the root system of $A(k)$ forms a Riesz basis of H?

Let us explain the root system. Let A be a linear operator on H $A\phi = \lambda\phi$, $\phi \neq 0$. Consider equation $A\phi_1 - \lambda\phi_1 = \phi$. If this equation is solvable, ϕ_1 is called a root vector of A corresponding to eigenvalue λ and eigenvector ϕ. If ϕ_1 exists consider equations $A\phi_k - \lambda\phi_k = \phi_{k-1}$, $k > 1$. It is known [7] that only a finite number r of root vectors $\phi_1, \ldots, \phi_r$ associated with ϕ exist. The chain$(\phi, \phi_1, \ldots, \phi_r)$ is called a Jordan chain with the length $r + 1$. The union of all root vectors of a linear operator A corresponding to all eigenvalues of A is called the root system of A. It is well known from linear algebra that the eigensystem of a nonselfadjoint operator may not form a basis. For example if the operator A is an operator in $\mathbb{R}^2$ with the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then A has only one eigenvector so that the eigensystem of A does not form a basis of $\mathbb{R}^2$. It is also known that the root system of any linear operator (matrix) in $\mathbb{R}^n$ forms a basis of $\mathbb{R}^n$. Of course in $\mathbb{R}^n$ any basis is Riesz basis. In a Hilbert space (infinite dimensional space) this is not true. For practice it is important to have affirmative answers for questions 1) and 3). Indeed, if the eigensystem of $A(k)$ does not form a Riesz basis but its root system forms a Riesz basis of H, then it is still possible to solve equation (6) using the root system of A.

2. What is SEM?

In order to explain what SEM is, consider the problem

$$u_{tt} - \Delta u = 0 \quad \text{in } \Omega; \quad u|_\Gamma = 0; \quad u|_{t=0} = 0, \; u_t|_{t=0} = f(x), \tag{12}$$

where $f \in C_0^\infty$ (a smooth function which vanishes for large $|x|$.

If $G(x,y,-p^2)$ is the Green function of the problem

$$(-\Delta + p^2)G = \delta(x - y) \text{ in } \Omega, \quad G|_\Gamma = 0, \quad \operatorname{Re} p > 0 \tag{13}$$

then the solution of (12) can be written as

$$u = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \exp(pt)\, \bar{u}(x,p)\, dp, \tag{14}$$

where

$$\overline{u}(x,p) = \int_{\Omega} G(x,y,-p^2) f(y) dy \tag{15}$$

Suppose that $\overline{u}(x,p)$ is meromorphic in p on the whole complex plane p (this is actually true [8]-[12]) and suppose the following estimate is valid

$$|\overline{u}(x,p)| \leq \frac{c}{1+|p|^a}, \quad a > 1/2, \quad c = \text{const.} > 0, \quad \text{Re}\,p > -A, \quad |\text{Im}\,p| > N_A, \tag{16}$$

where $A > 0$ is arbitrary.

This estimate follows from the Lax-Phillips result [12] and the arguments given in [9], [10], [13]. Under the assumptions made, the contour of integration in (14) can be moved to the left and one gets

$$u = \sum_{j=1}^{N} c_j t^{b_j - 1} \exp(p_j t) + o(\exp(-\text{Re}\, p_N t)) \tag{17}$$

where p_j are the poles of $G(x, y, -p^2)$, b_j are the multiplicity of the pole p_j. Expansion (17) is called the SEM expansion. Actually such expansions were known for a long time for various concrete problems of mathematical physics (especially in cases when the solution can be represented explicitly in the form of series). The main difficulty is to prove estimate (16) which allows us to move the contour on integration in (14). If only the mermorphic nature of $\overline{u}(x,p)$ as a function of p

is established, then no [illegible] expansions of the type (17) can be proved in general because there is a possibility that poles $-\varepsilon_n + i\omega_n$ with very small $\varepsilon_n > 0$ and very large ω_n can exist. The general Mittag-Löffler representation can not be applied for derivation of SEM expansion (17), because this representation uses special system of expanding contours, while the derivation of (17) requires the possibility to move our particular contour $(c - i\infty, c + i\infty)$ to the left. From a practical point of view, the SEM expansion is used at present according to the following scheme:

Suppose that only a few terms in (17) are essential, e.g. 1-3. This will be true if $|\mathrm{Re}\, p_j| \gg \mathrm{Re} p_3$ for $j > 3$. Then in experiments the transient field $u(x,t)$ is measured and each p_j, $j = 1,2,3$ is determined. It is assumed that the location of these complex poles of the Green function $G(x,y,-p^2)$ can give information enough to identify the obstacle (the scatterer D). This assumption has not been backed theoretically. Nevertheless, if there is a finite set of scatters (say flying targets) it is possible to believe that a one to one correspondence can be established empirically between the scatterers and the corresponding complex poles.

An interesting inverse problem can be formulated in connection with this question:

Inverse problem: Given a set of complex numbers $\{p_j\}$, $\mathrm{Re}\, p_j < 0$, is it possible to find [illegible] the Green function [illegible] poles $\{p_j\}$? Does the set [illegible]

What restrictions must be imposed on the set $\{p_j\}$, $\mathrm{Re}\, p_j < 0$ in order that this set will be the set of complex poles of the Green function of a scatterer?

If the scatterer is a star-like body and the boundary condition is the Dirichlet condition then the set $\{p_j\}$ must satisfy the condition $|\mathrm{Re}\, p_j| > a \ \ln |\mathrm{Im}\, p_j| + b$, $a > 0$ [12]. It seems that no other information on the problem is available.

From a practical point of view this problem may not be so important as it seems. First, only a few complex poles are available. It seems hopeless to make any general conclusions about the scaterer from this information without severe restrictions on the set of scatterers. (For example, if it is apriori known that the scatterer is a ball, it is possible to determine its radius from the above information.) That is why this author thinks that from a practical point of view in order to use the SEM for identification of scatterers it is more useful to work out tables of responses of the typical scatterers, then to try to develop a theory of the posed (which is very interesting from a theoretical point of view) inverse problem.

3. The following questions arise naturally in connection with the EEM and SEM methods:

1) Does the root system of the integral operators in diffraction theory ([illegible]) form a Riesz basis of H?

2) [illegible] coincide with the eigen-

3) Do the complex poles of the Green function depend continuously on the obstacle? In more detail: suppose that $x_j = x_j(t_1,t_2)$, $1 \le t_1,t_2 \le 1$ are parametric equations of Γ, $y_j = x_j(t_1,t_2) + \varepsilon z_j(t_1,t_2)$, $0 \le t_1,t_2 \le 1$, $0 < \varepsilon \le 1$ are parametric equations of the surface of perturbed scatterer. Let us assume that $x_j(t)$, $z_j(t) \in C^2(\Delta)$, $t = t_1,t_2$, $\Delta = \{t_i \le t_1,t_2 \le 1\}$. Let us fix an arbitrary number $R > 0$ and let p_j, $1 \le j \le r(R)$ be the complex poles of the unperturbed Green function which lie in the circle $|p_j| \le R$. Let $p_j(\varepsilon)$ be the complex poles of the perturbed Green function. Our question can now be formulated as follows: is it true that $p(\varepsilon) \to p_j$, as $\varepsilon \to 0$, uniformly in $1 \le j \le n(R)$ provided that the numeration of $p_j(\varepsilon)$ is properly done?

4) How can one calculate the complex poles?

5) What are sufficient conditions for the validity of SEM expansion (17)?

6) Is it possible to calculate complex poles via calculation of zeros of some functions?

§3. What has been rigorously established in EEM and SEM methods?

In this section we give answers to questions 1)-6) of Section 2.3. No proofs will be given but the results obtained will be formulated and references will be given. Proofs are omitted for three reasons: 1) they are difficult for engineers, 2) they are long, and 3) they can found in the cited papers.

1. In order to formulate the answer to question 1) of Section 2.3 we must explain what a Riesz basis with brackets is. Let $\{f_j\}$ be an orthonormal basis of H, $\{h_j\}$ be a complete and minimal system in H. (A complete system $\{h_j\}$ is called minimal if the system $\{h_j\} \setminus h_k$ is not complete for any k, $k = 1,2,3\ldots$. In other words if we remove any element h_k of our system, we obtain an incomplete system). Let $m_1 < m_2 < \cdots$ be an arbitrary increasing sequence of integers; F_j is the linear space with the basis $\{f_{m_{j-1}}, f_{m_{j-1}+1}, \ldots, f_{m_j-1}\}$; H_j is the linear space with the basis $\{h_{m_{j-1}}, \ldots, h_{m_j-1}\}$. Suppose that there exists a linear bounded operator B with bounded B^{-1}, defined on all H, such that $BH_j = F_j$, $j = 1,2,\ldots$. Then the system $\{h_j\}$ is called a Riesz basis of H with brackets. This definition is equivalent (see [14]) to the following. Let P_j be projectors in H onto H_j. Suppose that for any $f \in H$,

$$c_1 \|f^2\| \leq \sum_{j=1}^{\infty} \|P_j f\|^2 \leq c_2 \|f\|^2, \quad c_1 > 0 ,$$

then the system $\{h_j\}$ is called a Riesz basis of H with brackets.

It is proved that the root system of operator $A(k)$ (see formula (6)) forms a Riesz basis with brackets [6]. The same is true for the operator arising in the exterior Neumann boundary value problem [1], [6]. The same is also true for the electrodynamics scattering problem [1].

2. If the integral operator of the diffraction problem is normal then its eigensystem coincides with its root system. This is the case in the problem (1)-(4) for a spherical surface Γ and for a linear antenna. First this was observed in [5]. An operator is called normal if $A^*A = AA^*$, where A^* is the adjoint operator. The condition $AA^* = A^*A$ is the condition on Γ provided that the kernel of the integral operator A is given. (see [5] for details).

3. The answer to question 3 from section 2.3 is affirmative (see [6] for details).

4. A general method (with the proof of its convergence) for calculating the complex poles of the Green's functions in diffraction and potential scattering theory was given in [4], [15] (also see [6]). The method can be explained for the problem with the impedance boundary condition

$$(\nabla^2 + k^2)u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial N} - h\,u\Big|_{\Gamma} = 0 \tag{18}$$

where $h = \text{const.}$, $\mathrm{Re}\, h > 0$, N is the outer normal to Γ. We look for a solution of the problem without sources in the form

$$u = \int_\Gamma \frac{\exp(ik|x-t|)}{4\pi|x-t|} \sigma(t)\, dt \;. \tag{19}$$

From (19) and (18) it follows that

$$\sigma = Q\sigma \;, \tag{20}$$

where

$$Q\sigma = \int_\Gamma \frac{\partial}{\partial N_t} \frac{\exp(ikr_{st})}{2\pi r_{st}} \sigma(s)ds - h \int_\Gamma \frac{\exp(ikr_{st})}{2\pi r_{st}} \sigma(s)ds \tag{21}$$

Let $\{\phi_j\}$ be a Riesz basis of $H = L^2(\Gamma)$,

$$\sigma_N = \sum_{j=1}^{N} c_j \phi_j \;. \tag{22}$$

Substituting (22) in (20) and multiplying by ϕ_i in H we obtain:

$$\sum_{j=1}^{N} b_{ij}(k)c_j = 0, \quad 1 \le i \le N \;; \quad b_{ij}(k) = \delta_{ij} - (Q\phi_j, \phi_i). \tag{23}$$

This system has nontrivial solutions if and only if

$$\det b_{ij}(k) = 0. \tag{24}$$

The left-hand side of (24) is an entire function of k. Let $k_m^{(N)}$ be its roots. Then there exist the limits $\lim_{N\to\infty} k_m^{(N)} = k_m$ and k_m are the poles of the Green function corresponding to the problem (18). Moreover, all the complex poles can be obtained by this method. Proofs are given in [4], [15], [6]. From a practical point of view there are two nontrivial points in performing this method. 1) calculation of $b_{ij}(k)$ by formula (23) and 2) numerical solution of equation (24). For both steps there are methods available in the literature on numerical analysis.

5. Sufficient conditions for the validity of SEM expansion (17) were given in Section 2.

6. The set of the complex poles of the Green function of the problem (1) - (4) coincides with the set of the complex zeros of the eigenvalues $\rho_n(k)$ of the operator $A(k)$:

$$A(k)\phi_n = \rho_n(k)\phi_n, \quad n = 1,2,\ldots \quad . \tag{25}$$

Indeed, let $G = \frac{R(x,y)}{(k-z)^r} + \ldots$, i.e. z is a pole of the Green function G, $G|_\Gamma = 0$,

$$G = g - \int_\Gamma g(x,t,1)\mu\ (t,y,k)dt, \quad g = \frac{\exp(ikr_{xy})}{4\pi r_{xy}}, \tag{26}$$

$$\mu = \frac{\partial G}{\partial N_t} .$$

Multiplying (26) by $(k - z)^r$ and taking $k \to z$, we get

$$\int_\Gamma g(x,t,z)\ \frac{\partial R(t,y)}{\partial N_t}\ dt = 0, \qquad x \in \Gamma . \tag{27}$$

The kernel $R(t,y)$ is degenerate. Thus a function $\phi \not\equiv 0$ exists such that

$$\int_\Gamma g(x,t;z)\ \phi(t)dt = 0 \tag{28}$$

This means that $k = z$ is a zero of some of the functions $\lambda_n(k)$. Conversely, if $\phi \neq 0$ is a solution of (28), then

$$u = \int_{\Gamma} g(x,t;z)\ \phi(t)\,dt \tag{29}$$

is a solution of the problem

$$(\nabla^2 + z^2)u = 0 \quad \text{in } \Omega, \quad u|_{\Gamma} = 0 \tag{30}$$

with the outgoing asymptotic at infinity. Hence $u \equiv 0$ in Ω if z is not a pole of G. Since z^2 is complex, $u(x) \equiv 0$ in D (as a solution of the homogeneous interior problem). By the boundary value jump relation $\phi \equiv 0$. This contradiction proves that z is a pole of G. A variational method for calculation of eigenvalues of nonselfadjoint compact operators is given in [6].

§4. Open problems.

1) The inverse problem formulated in Section 2 is of interest. It is very interesting to have partial answers: what information about the geometry of a scatterer can be obtained from the location of the complex poles.

2) There is a conjecture [3] that the complex poles of the Green function of the problem (1)-(4) for a convex smooth compact boundary are simple. It would be interesting to prove it or to give a counterexample.

3) It would be interesting to perform numerically method described in Section 3.6 in some practical problems.

4) In [16] some properties of the purely real poles $\mathrm{Re}\, p_j < 0$, $\mathrm{Im}\, p_j = 0$ were established. It would be interesting to tell what information about the geometry of an obstacle can be obtained from the location of the purely real poles. In the physics literature the complex plane $k = ip$ is usually used. On this plane the purely real complex poles are purely imaginary, $\mathrm{Re}\, k_j = 0$, $\mathrm{Im}\, k_j < 0$.

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